4, 5, 9, will then have the same sum, and the second members in each square will be similarly related. The square is completed by filling the remaining rows with replicas and turning over a central diagonal. Fig. 10 is a square of order 16 constructed from the outline Fig. 9. It has all the properties of the 16^2 shown in Fig. 6, and is also magic on its 64 knight paths.

The following is an arrangement of the couplets for a square of order 24:

1.24–4.21	8.17–5.20	10.15–13.12	11.14–16.9	22.3–18.7	23.2–19.6

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ORNATE MAGIC SQUARES OF COMPOSITE ODD ORDERS.

When we consider these orders in the light of the general rule used for orders $\equiv 0$ (mod. 4) it appears at first sight that they cannot be made to fulfil all the conditions; but it is not essential to the *ply* property, nor to the balanced magic subsquares that the numbers be taken in complementary pairs for the subsquares of the outline. All that is necessary is that the groups of numbers chosen shall all have the same sum.

Suppose, as an illustration, we are dealing with order 15. If we can arrange the first 15 natural numbers in five balanced

2	7	15
7	15	2
15	2	7

Fig. 1.

2	6	12	11	9
15	13	8	3	1
7	5	4	10	14

Fig. 2.

columns, three in a column, and form five magic outlines of order 3, using a different column thrice repeated for each outline, we shall have five balanced magic outlines like Fig. 1. These can be arranged in the first row of subsquares with replicas in the following rows. If we can turn this outline upon itself in some way to avoid repetitions, we shall have a magic square which will be 9-ply and with magic subsquares. But will it be pandiagonal?

In the small outlines of 9 cells made from Fig. 1 as a pattern, it

will be noticed that like numbers must always occur in parallel diagonals; therefore if we arrange the five small squares so that like numbers always lie along / diagonals, the great outline will be "boxed" and therefore magic in \ diagonals, but in the / diagonals we shall have in every case only five different numbers each occurring thrice. The problem is thus reduced to finding a magic rectangle 3×5. We therefore construct such a rectangle by the method of "Complementary Differences"[1] as shown in Fig. 2.

In Fig. 3 we have the five magic outlines constructed from the five columns of the rectangle, and placed side by side with like

2	7	15	6	5	13	12	4	8	11	10	3	9	14	1
7	15	2	5	13	6	4	8	12	10	3	11	14	1	9
15	2	7	13	6	5	8	12	4	3	11	10	1	9	14

Fig. 3.

2	12	9	6	11	15	8	1	13	3	7	4	14	5	10
9	6	11	2	12	1	13	3	15	8	14	5	10	7	4
11	2	12	9	6	3	15	8	1	13	10	7	4	14	5
12	9	6	11	2	8	1	13	3	15	4	14	5	10	7
6	11	2	12	9	13	3	15	8	1	5	10	7	4	14

Fig. 4.

numbers always in the / diagonals, and so disposed that *the number* in any / diagonal is always succeeded (when the diagonal passes across into a neighboring square) by the number which succeeds it in its row in the rectangle.

If an associated square is required the magic rectangle must be associated and the large rectangle of subsquares must also be associated as a whole. It will be noticed that all these conditions will be fulfilled in practice if we write the successive columns of the magic rectangle Fig. 2 along the \ central diagonals of the successive square outlines in the larger rectangle Fig. 3, and fill in all the / diagonals with replicas. If now all the remaining rows of

[1] See "The Construction of Magic Squares and Rectangles by the Method of Complementary Differences," by W. S. Andrews, *Monist*, July, 1910, Vol. XX, No. 3.

subsquares be filled with replicas of the top row it will be found that the whole outline *cannot* be turned over either of its central diagonals without repetitions in the magic, but it *can* be turned successfully *in its own plane,* about its central point through one right angle, without repetitions. (This will bring the top row in coincidence with the left-hand column, so that the right-hand square in Fig. 3 is turned on its side and lies over the left-hand square.) The resulting magic is shown in Fig. 6. It is magic on its 15 rows,

155	28	171	125	88	156	20	178	126	80	163	21	170	133	81
44	211	114	14	181	39	224	106	9	194	31	219	119	1	189
139	98	57	199	68	147	94	53	207	64	143	102	49	203	72
157	30	167	127	90	152	22	180	122	82	165	17	172	135	77
40	213	116	10	183	41	220	108	11	190	33	221	115	3	191
140	103	51	200	73	141	95	58	201	65	148	96	50	208	66
164	16	174	134	76	159	29	166	129	89	151	24	179	121	84
34	218	117	4	188	42	214	113	12	184	38	222	109	8	192
142	105	47	202	75	137	97	60	197	67	150	92	52	210	62
160	18	176	130	78	161	25	168	131	85	153	26	175	123	86
35	223	111	5	193	36	215	118	6	185	43	216	110	13	186
149	91	54	209	61	144	104	46	204	74	136	99	59	196	69
154	23	177	124	83	162	19	173	132	79	158	27	169	128	87
37	225	107	7	195	32	217	120	2	187	45	212	112	15	182
145	93	56	205	63	146	100	48	206	70	138	101	55	198	71

Fig. 5. S = 1695

15 columns, 30 diagonals and 60 knight paths, also 9-ply and associated. The 25 subsquares of order 3 all sum 339 on their 3 rows and 3 columns. (It is easy to see that only one of them can have magic central diagonals, for a magic of order 3 can only have this property when it is associated, and in this case the mean number must occupy the central cell, but there is here only one mean number, viz., 113, therefore only the central subsquare can have magic diagonals.)

In exactly the same manner as above described, by using the

long rows of the magic rectangle Fig. 2, instead of the short columns, we can construct another ornate magic of order 15.

Fig. 4 shows the first row of 25-celled subsquares constructed from the *rows* of the rectangle, and using a magic square of order 5 as pattern. If we fill the two remaining rows of subsquares with replicas the outline can be turned over either of its central diagonals. The resulting square is shown in Fig. 7. It is magic on 15 rows, 15 columns, 30 diagonals and 60 knight paths, also 25-ply and asso-

2	127	210	6	125	208	12	124	203	11	130	198	9	134	196
202	15	122	200	13	126	199	8	132	205	3	131	209	1	129
135	197	7	133	201	5	128	207	4	123	206	10	121	204	14
32	157	150	36	155	148	42	154	143	41	160	138	39	164	136
142	45	152	140	43	156	139	38	162	145	33	161	149	31	159
165	137	37	163	141	35	158	147	34	153	146	40	151	144	44
107	172	60	111	170	58	117	169	53	116	175	48	114	179	46
52	120	167	50	118	171	49	113	177	55	108	176	59	106	174
180	47	112	178	51	110	173	57	109	168	56	115	166	54	119
182	82	75	186	80	73	192	79	68	191	85	63	189	89	61
67	195	77	65	193	81	64	188	87	70	183	86	74	181	84
90	62	187	88	66	185	83	72	184	78	71	190	76	69	194
212	22	105	216	20	103	222	19	98	221	25	93	219	29	91
97	225	17	95	223	21	94	218	27	100	213	26	104	211	24
30	92	217	28	96	215	23	102	214	18	101	220	16	99	224

Fig. 6 S = 1695

ciated. Also the nine subsquares of order 5 are balanced nasiks, summing 565 on their 5 rows, 5 columns and 10 diagonals.

The above method can of course be used when the order is the square of an odd number, e. g., orders 9, 25, etc. These have previously been dealt with by a simpler method which is not applicable when the order is the product of different odd numbers.

A similar distinction arises in the case of orders $\equiv 0 \pmod{4}$ previously considered. These were first constructed by a rule which applied only to orders of form 2^m, e. g., 4, 8, 16, 32, etc., but the general rule is effective in every case.

There are two other ornate squares of order 15, shown in Figs. 5 and 8, these four forms of ornate squares being numbered in ascending order of difficulty in construction. Fig. 5 is constructed by using the paths $\begin{Bmatrix} 3, & 5 \\ 5, & 3 \end{Bmatrix}$ and taking the period from the *continuous diagonal* of the magic rectangle Fig. 2.

Fig. 5 is magic on 15 rows, 15 columns, 30 diagonals, 60 knight paths, and is 9-ply, 25-ply and associated.

The square shown in Fig. 8 has been only recently obtained;

17	132	159	171	86	30	128	151	178	78	22	124	164	170	85
174	81	26	122	162	166	88	18	135	158	179	80	25	127	154
131	152	177	84	21	123	165	173	76	28	130	157	169	89	20
87	24	126	161	167	83	16	133	153	180	79	29	125	160	172
156	176	77	27	129	163	168	90	23	121	155	175	82	19	134
212	12	39	111	191	225	8	31	118	183	217	4	44	110	190
114	186	221	2	42	106	193	213	15	38	119	185	220	7	34
11	32	117	189	216	3	45	113	181	223	10	37	109	194	215
192	219	6	41	107	188	211	13	33	120	184	224	5	40	112
36	116	182	222	9	43	108	195	218	1	35	115	187	214	14
92	207	144	51	71	105	203	136	58	63	97	199	149	50	70
54	66	101	197	147	46	73	93	210	143	59	65	100	202	139
206	137	57	69	96	198	150	53	61	103	205	142	49	74	95
72	99	201	146	47	68	91	208	138	60	64	104	200	145	52
141	56	62	102	204	148	48	75	98	196	140	55	67	94	209

Fig. 7. $S = 1695$

for many years the conditions therein fulfilled were believed to be impossible. It is magic on 15 rows, 15 columns and 30 diagonals, and is 3×5 rectangular ply, i. e., *any* rectangle 3×5 with long axis horizontal contains numbers whose sum is the magic sum of the square. Also the 15 subrectangles are balanced magics, summing 565 in their three long rows and 339 in their five short columns. This square is not associated, and only half of its knight paths are magic.

The three squares of order 15, shown in Figs. 5, 6 and 7, are

described as magic on their 60 knight paths, but actually they are higher nasiks of Class II, as defined at the end of my pamphlet on *The Theory of Path Nasiks.*[2] Further, the squares in Figs. 6 and 7 have the following additional properties.

Referring to the square in Fig. 7 showing subsquares of order 5; if we superpose the diagonals of these subsquares in the manner described in my paper on "Fourfold Magics" (*The Monist,* Vol. XX, p. 618, last paragraph), we obtain three magic parallelopipeds

37	93	191	81	163	32	99	185	89	160	45	102	188	79	151
167	219	5	59	115	180	222	8	49	106	172	213	11	51	118
135	27	143	199	61	127	18	146	201	73	122	24	140	209	70
97	183	86	156	43	92	189	80	164	40	105	192	83	154	31
212	9	50	119	175	225	12	53	109	166	217	3	56	111	178
30	147	203	64	121	22	138	206	66	133	17	144	200	74	130
187	78	161	36	103	182	84	155	44	100	195	87	158	34	91
2	54	110	179	220	15	57	113	169	211	7	48	116	171	223
150	207	68	124	16	142	198	71	126	28	137	204	65	134	25
82	153	41	96	193	77	159	35	104	190	90	162	38	94	181
47	114	170	224	10	60	117	173	214	1	52	108	176	216	13
210	72	128	19	136	202	63	131	21	148	197	69	125	29	145
157	33	101	186	88	152	39	95	194	85	165	42	98	184	76
107	174	215	14	55	120	177	218	4	46	112	168	221	6	58
75	132	23	139	195	67	123	26	141	208	62	129	20	149	205

Fig. 8. S = 1695

$5 \times 5 \times 3$. Denoting each subsquare by the number in its central cell, the three parallelopipeds will be:

I.	53,	169,	117.
II.	177,	113,	49.
III.	109,	57,	173.

These three together form an octahedroid $5 \times 5 \times 3 \times 3$ which is associated and magic in each of the four directions parallel to its edges.

If we deal in like manner with Fig. 6 which has subsquares of

order 3 we obtain five magic parallelopipeds of order $3\times3\times5$ together forming an associated magic octahedroid of order $3\times3\times5\times5$. Since the lengths of the edges are the same as those of the octahedroid formed from Fig. 7 square, these two four-dimensional figures are identical but the distribution of the numbers in their cells is not the same. They can however be made completely identical both in form and distribution of numbers by a slight change in our method of dealing with the square Fig. 6, i. e., by taking the square plates to form the parallelopipeds from the knight paths instead of the diagonals. Using the path $-1, 2$ we get 225, 106, 3, 188, 43 for the first plates of each parallelopiped, and then using $2, -1$ for the successive plates of each, we obtain the parallelopipeds:

I.	225,	8,	31,	118,	183
II.	106,	193,	213,	15,	38
III.	3,	45,	113,	181,	223
IV.	188,	211,	13,	33,	120
V.	43,	108,	195,	218,	1

This octahedroid is completely identical with that previously obtained from Fig. 7, as can be easily verified by taking any number at random and writing down the four series of numbers through its containing cell parallel to the edges, first in one octahedroid and then in the other. The sets so obtained will be found identical.

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PANDIAGONAL-CONCENTRIC MAGIC SQUARES OF ORDERS 4*m*.

These squares are composed of a central pandiagonal square surrounded by one or more bands of numbers, each band, together with its enclosed numbers, forming a pandiagonal magic square.

The squares described here are of orders 4*m* and the bands or borders are composed of double strings of numbers. The central square and bands are constructed simultaneously instead of by the usual method of first forming the nucleus square and arranging the bands successively around it.

[2] *The Theory of Path Nasiks*, by C. Planck, M.A., M.R.C.S., printed by A. J. Lawrence, Rugby, Eng.